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q-FRACTIONAL DIRAC TYPE SYSTEMS

Bilender P. Allahverdiev and Hüseyin Tuna

Abstract. This paper is devoted to study a regular q-fractional Dirac type system. We investigate the properties of the eigenvalues and the eigenfunctions of this system. By using a fixed point theorem we give a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions.

1. Introduction

q-calculus deals with the investigation and applications of quantum derivatives and quantum integrals. It is an interesting topic having interconnections with various problems of mathematical physics and quantum mechanics ([8–10, 12, 14, 16, 17, 23, 24, 30]). For the q-calculus, we refer the reader to the books [7, 13, 18].

The fractional q-calculus is the generalization of the q-calculus. In the recent years, some results have been derived in q-fractional equations [5–7, 15, 20–22, 25, 26]. Mansour [25] introduced q-fractional Sturm-Liouville problems containing the left-sided Caputo q-fractional derivative and the right-sided Riemann-Liouville q-fractional derivative. The author used a fixed point theorem to introduce a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions of q-fractional Sturm-Liouville problems. Al-Towailb studied the regular q-fractional Sturm-Liouville problems. The author proved properties of the eigenvalues and the eigenfunctions in [5]. In [26], the author introduced the essential q-fractional variational analysis needed in proving the existence of a countable set of real eigenvalues and associated orthogonal eigenfunctions for the regular q-fractional Sturm-Liouville problems. Allahverdiev and Tuna [3] proved a theorem on the completeness of the system of eigenvectors and associated vectors of the dissipative q-fractional Sturm-Liouville operators.

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It is well known that the Dirac systems defined by

$$\begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}+\begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \lambda\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \tag{1.1}$$

where $x \in [a,b]$, play an important role in relativistic quantum mechanics. These systems describe spin 1/2 particles, including electrons, neutrinos, muons, protons, neutrons, quarks, and their corresponding anti-particles. For the history and details of the Dirac systems, see [11, 19, 29, 31] and their references. In this paper, we are interest in a $q-$fractional version of the system (1.1) defined by

$$\begin{pmatrix} 0 & -\mathcal{D}^{\alpha}_{q,a^-} \\ {}^{c}\mathcal{D}^{\alpha}_{q,0^+} & 0 \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}+\begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \lambda\begin{pmatrix} \omega_1(x) & 0 \\ 0 & \omega_2(x) \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

where $x \in (0,a)$. To the best of the authors' knowledge there are no results available in the literature considering this system. These results are a generalization of the regular q-Dirac system introduced in [2].

2. Preliminaries

First of all, we recall the notations and some basic properties for q-fractional calculus theory, which are useful in the following discussion (see [1, 4, 7, 13, 18, 25, 27, 28]). Throughout this paper, we assume that $0 < q < 1$ and A is a q-geometric set, i.e., $qx \in A$ whenever $x \in A$. For every $t > 0$, we define the sets $A_{t,q}$, $A^*_{t,q}$ and $\mathcal{A}_{t,q}$, respectively, by

$$A_{t,q} := \{tq^n : n \in \mathbb{N}\}, \ A^*_{t,q} := A_{t,q} \cup \{0\},$$

and

$$\mathcal{A}_{t,q} := \{\pm tq^n : n \in \mathbb{N}\}.$$

Let $y(.)$ be a complex-valued function on A. The *q-difference operator* $\mathcal{D}_q$ is defined by

$$\mathcal{D}_q y(x) = \frac{y(qx) - y(x)}{(q-1)x} \text{ for all } x \in A\backslash\{0\}.$$

The q-derivative at zero is defined by

$$\mathcal{D}_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on x. A *right-inverse* to D_q, the *Jackson* q*-integration* is given by

$$\int_0^x f(t)\, d_q t = x(1-q)\sum_{n=0}^{\infty} q^n f(q^n x) \ \ (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t \ \ (a, b \in A).$$

Let $L_q^2(0,a)$ be the space of all complex-valued functions defined on $[0,a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|^2\, d_q x\right)^{1/2} < \infty.$$

$L_q^2(0,a)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_0^a f(x)\,\overline{g(x)} d_q x, \quad f, g \in L_q^2(0,a),$$

and the orthonormal basis

$$\phi_n(x) = \begin{cases} \frac{1}{\sqrt{x(1-q)}}, & x = aq^n, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 0, 1, 2, ...$(see [7]).

Definition 2.1. *A function f which is defined on A, $0 \in A$, is said to be q-regular at zero if*

$$\lim_{n\to\infty} f(xq^n) = f(0)$$

for every $x \in A$ (see [7]).

Let $C(A)$ denote the space of all q-regular at zero functions on A. This space is a normed space with the norm function

$$\|f\| = \sup\left\{|f(xq^n)|,\ x \in A,\ n \in \mathbb{N}\right\}.$$

(see [7]).

Definition 2.2. *A q-regular at zero function f which is defined on $A_{t,q}^*$ is said to be q-absolutely continuous if*

$$\sum_{j=0}^{\infty} \left|f\left(uq^j\right) - f\left(uq^{j+1}\right)\right| \leq K,\ \forall u \in A_{t,q}^*,$$

for K is a constant depending on the function f (see [7]).

The space of all q-absolutely continuous functions on $A_{t,q}^{*}$ is denoted by $AC_q\left(A_{t,q}^{*}\right)$. Note that $\mathcal{A}C_q(A_{q,t}^{*})\subseteq C(A_{q,t}^{*})$.

For $n\in\mathbb{N}$ and $\alpha, a_1, .., a_n\in\mathbb{C}$; the *$q$-shifted factorial*, the *multiple q-shifted factorial* and the *q-binomial coefficients* are defined by

$$(a;q)_0=1,\ (a;q)_n=\prod_{k=0}^{n-1}\left(1-aq^k\right),\ (a;q)_\infty=\prod_{k=0}^{\infty}\left(1-aq^k\right),$$

$$(a_1,a_2,...,a_k:q)=\prod_{j=1}^{k}(a_j;q)_n$$

and

$$\begin{bmatrix}\alpha\\0\end{bmatrix}_q=1,\ \begin{bmatrix}\alpha\\n\end{bmatrix}_q=\frac{(1-q^\alpha)\left(1-q^{\alpha-1}\right)...\left(1-q^{\alpha-n+1}\right)}{(q;q)_n},$$

respectively (see [7]). The *generalized q-shifted factorial* is defined by

$$(a;q)_\nu=\frac{(a;q)_\infty}{(aq^\nu;q)_\infty}\ (\nu\in\mathbb{R})$$

(see [7]). The *q-Gamma function* is defined by

$$\Gamma_q(z)=\frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z},\ z\in\mathbb{C},\ |q|<1$$

(see [7]).

Definition 2.3. *Let $0<\alpha\leq 1$. The left-sided and right-sided Riemann-Liouville q-fractional operator are given by the formulas*

$$\mathcal{I}_{q,a^+}^{\alpha}f(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x\left(\frac{qt}{x};q\right)_{\alpha-1}f(t)\,d_qt, \tag{2.1}$$

$$\mathcal{I}_{q,b^-}^{\alpha}f(x)=\frac{1}{\Gamma_q(\alpha)}\int_{qx}^{b}t^{\alpha-1}\left(\frac{qx}{t};q\right)_{\alpha-1}f(t)\,d_qt, \tag{2.2}$$

respectively (see [25]).

Definition 2.4. *Let $\alpha>0$ and $\lceil\alpha\rceil=m$. The left-sided and right-sided Riemann-Liouville fractional q-derivatives of the order α are defined, respectively, as follows:*

$$\mathcal{D}_{q,a^+}^{\alpha}f(x)=\mathcal{D}_q^m\mathcal{I}_{q,a^+}^{m-\alpha}f(x), \tag{2.3}$$

$$\mathcal{D}_{q,b^-}^{\alpha}f(x)=\left(\frac{-1}{q}\right)^m\mathcal{D}_{q^{-1}}^m\mathcal{I}_{q,b^-}^{m-\alpha}f(x). \tag{2.4}$$

Similar formulas give the left-sided and right-sided Caputo fractional q-derivatives of order α, respectively as follows:

$$ {}^{c}\mathcal{D}_{q,a^{+}}^{\alpha}f(x)=\mathcal{I}_{q,a^{+}}^{m-\alpha}\mathcal{D}_{q}^{m}f(x), $$
$$ {}^{c}\mathcal{D}_{q,b^{-}}^{\alpha}f(x)=\left(\frac{-1}{q}\right)^{m}\mathcal{I}_{q,b^{-}}^{m-\alpha}\mathcal{D}_{q^{-1}}^{m}f(x) $$

(see [25]).

In order to prove the main results, we also need the following lemmas. One can find them in [25].

Lemma 2.5.

i) The left-sided Riemann-Liouville q-fractional operator satisfies the semi-group property

$$ \mathcal{I}_{q,a^{+}}^{\alpha}\mathcal{I}_{q,a^{+}}^{\beta}=\mathcal{I}_{q,a^{+}}^{\alpha+\beta}f(x),\ x\in A_{q,a}^{*}, \tag{2.5} $$

for any function defined on $A_{q,a}$ and for any values of α and β.

ii) The right-sided Riemann-Liouville q-fractional operator satisfies the semi-group property

$$ \mathcal{I}_{q,b^{-}}^{\alpha}\mathcal{I}_{q,b^{-}}^{\beta}f(x)=\mathcal{I}_{q,b^{-}}^{\alpha+\beta}f(x),\ x\in A_{q,b}^{*}, $$

for any function defined on $A_{q,b}$ and for any values of α and β .

Lemma 2.6. *Let $\alpha\in(0,1)$.*

i) If $f\in L_{q}^{1}\left(A_{q,a}^{*}\right)$ such that $\mathcal{I}_{q,0^{+}}^{\alpha}f\in AC_{q}\left(A_{t,q}^{*}\right)$ then

$$ {}^{c}\mathcal{D}_{q,0^{+}}^{\alpha}\mathcal{I}_{q,0^{+}}^{\alpha}f(x)=f(x)-\frac{\mathcal{I}_{q,0^{+}}^{\alpha}f(0)}{\Gamma_{q}(1-\alpha)}x^{-\alpha}. $$

Moreover, if f is bounded on $A_{t,q}^{*}$ then

$$ {}^{c}\mathcal{D}_{q,0^{+}}^{\alpha}\mathcal{I}_{q,0^{+}}^{\alpha}f(x)=f(x). $$

ii) If $f\in L_{q}^{1}(A_{q,a})$ then

$$ \mathcal{D}_{q,0^{+}}^{\alpha}\mathcal{I}_{q,0^{+}}^{\alpha}f(x)=f(x). $$

iii) If f is a function defined on $A_{t,q}^{*}$ then

$$ {}^{c}\mathcal{D}_{q,a^{-}}^{\alpha}\mathcal{I}_{q,a^{-}}^{\alpha}f(x)=f(x)-\frac{a^{-\alpha}}{\Gamma_{q}(1-\alpha)}\left(\frac{qx}{a};q\right)_{-\alpha}\left(\mathcal{I}_{q,a^{-}}^{\alpha}f\right)\left(\frac{a}{q}\right), $$
$$ \mathcal{D}_{q,a^{-}}^{\alpha}\mathcal{I}_{q,a^{-}}^{\alpha}f(x)=f(x), $$
$$ \mathcal{I}_{q,a^{-}}^{\alpha}\mathcal{D}_{q,a^{-}}^{\alpha}f(x)=f(x)-\frac{a^{\alpha-1}}{\Gamma_{q}(\alpha)}\left(\frac{qx}{a};q\right)_{\alpha-1}\left(\mathcal{I}_{q,a^{-}}^{1-\alpha}f\right)\left(\frac{a}{q}\right). $$

iv) If $f\in AC_{q}\left(A_{t,q}^{*}\right)$ then

$$ \mathcal{I}_{q,0^{+}}^{\alpha}\,{}^{c}\mathcal{D}_{q,0^{+}}^{\alpha}f(x)=f(x)-f(0). $$

We denote by $L_{q,\omega}^{2}\left(A_{t,\alpha}^{*};E\right)$ $(E:=\mathbb{R}^{2})$ the Hilbert space which consists of vector-valued functions with inner product

$$(f,g):=\int_{0}^{a}f_{1}(x)\overline{g_{1}(x)}\omega_{1}(x)d_{q}x+\int_{0}^{a}f_{2}(x)\overline{g_{2}(x)}\omega_{2}(x)d_{q}x, \tag{2.6}$$

where $f(x)=\begin{pmatrix} f_{1}(x) \\ f_{2}(x)\end{pmatrix}$, $g(x)=\begin{pmatrix} g_{1}(x) \\ g_{2}(x)\end{pmatrix}$, $f_{i}(x),\ g_{i}(x),\ \omega_{i\alpha}(x)\ (i=1,2)$ are real-valued functions on $A_{t,\alpha}^{*}$ and $\omega_{i}(x)>0,\ \forall x\in A_{t,\alpha}^{*},\ (i=1,2)$.

3. q-Fractional Dirac Systems

In the present section, our goal is to study the q-fractional Dirac system which includes the right-sided Caputo and the left-sided Riemann-Liouville fractional derivatives of same order α. Throughout this section, we assume $\alpha\in(0,1)$.

Let

$$\begin{aligned}\tau_{q,\alpha}y&:=\begin{pmatrix} 0 & -\mathcal{D}_{q,a^{-}}^{\alpha} \\ {}^{c}\mathcal{D}_{q,0^{+}}^{\alpha} & 0\end{pmatrix}\begin{pmatrix} y_{1} \\ y_{2}\end{pmatrix}+\begin{pmatrix} p(x) & 0 \\ 0 & r(x)\end{pmatrix}\begin{pmatrix} y_{1} \\ y_{2}\end{pmatrix}\\ &=\begin{pmatrix} -\mathcal{D}_{q,a^{-}}^{\alpha}y_{2}+p(x)y_{1} \\ {}^{c}\mathcal{D}_{q,0^{+}}^{\alpha}y_{1}+r(x)y_{2}\end{pmatrix},\end{aligned}$$

where $y:=\begin{pmatrix} y_{1} \\ y_{2}\end{pmatrix}$. With this notation, we consider the q-fractional Dirac type system:

$$\tau_{q,\alpha}f_{\lambda}=\lambda\omega f_{\lambda},\ a\leq x\leq b<\infty, \tag{3.1}$$

where $f_{\lambda}=\begin{pmatrix} f_{\lambda1} \\ f_{\lambda2}\end{pmatrix}$, $p(.)$, $r(.)$ are real-valued functions defined in $A_{t,\alpha}^{*}$, $\omega(x)=\begin{pmatrix} \omega_{1}(x) & 0 \\ 0 & \omega_{2}(x)\end{pmatrix}$, $\omega_{i}(.)$ are real-valued functions defined in $A_{t,\alpha}^{*}$ and $\omega_{i\alpha}(x)>0$, $\forall x\in A_{t,\alpha}^{*},\ (i=1,2)$, λ is a complex eigenvalue parameter and boundary conditions

$$c_{11}f_{\lambda1}(0)+c_{12}\mathcal{I}_{q,a^{-}}^{1-\alpha}f_{\lambda2}(0)=0, \tag{3.2}$$

$$c_{21}f_{\lambda1}(a)+c_{22}\mathcal{I}_{q,a^{-}}^{1-\alpha}f_{\lambda2}\left(\frac{a}{q}\right)=0, \tag{3.3}$$

with $c_{11}^{2}+c_{12}^{2}\neq0$ and $c_{21}^{2}+c_{22}^{2}\neq0$.

To pass from the differential expression $T_{q,\alpha}:=\omega^{-1}\tau_{q,\alpha}$ to operators, we introduce the space $H\subseteq L_{q,\omega}^{2}\left(A_{t,\alpha}^{*};E\right)\cap C\left(A_{t,\alpha}^{*};E\right)$ which consists of all q-regular at zero functions satisfying the conditions (3.2) and (3.3) with inner product (2.6).

THEOREM 3.1. *The operator $T_{q,\alpha}$ generated by q-fractional Dirac type system (FD) defined by (3.1)-(3.3) is formally self-adjoint on H.*

PROOF. Let $u(.), z(.) \in H$. Then, we have

$$\begin{aligned}
(T_{q,\alpha}u, z) - (u, T_{q,\alpha}z) &= \int_0^a \left({}^c\mathcal{D}^\alpha_{q,0^+}u_1 + r(x)\,u_2\right)\overline{z_2}d_qx \\
&+ \int_0^a \left(-\mathcal{D}^\alpha_{q,a^-}u_2 + p(x)\,u_1\right)\overline{z_1}d_qx \\
&- \int_0^a u_2\overline{\left({}^c\mathcal{D}^\alpha_{q,0^+}z_1 + r(x)\,z_2\right)}d_qx \\
&- \int_0^a u_1\overline{\left(-\mathcal{D}^\alpha_{q,a^-}z_2 + p(x)\,z_1\right)}d_qx \\
&= \int_0^a \left({}^c\mathcal{D}^\alpha_{q,0^+}u_1\right)\overline{z_2}d_qx - \int_0^a \left(\mathcal{D}^\alpha_{q,a^-}u_2\right)\overline{z_1}d_qx \\
&- \int_0^a u_2\overline{\left({}^c\mathcal{D}^\alpha_{q,0^+}z_1\right)}d_qx + \int_0^a u_1\overline{\left(\mathcal{D}^\alpha_{q,a^-}z_2\right)}d_qx.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^a \left({}^c\mathcal{D}^\alpha_{q,0^+}u_1\right)\overline{z_2}d_qx &= \int_0^a u_1\overline{\left(-\mathcal{D}^\alpha_{q,a^-}z_1\right)}d_qx \\
-\left[u_1(a)\,\overline{\mathcal{I}^{1-\alpha}_{q,a^-}z_2\left(\frac{a}{q}\right)} - u_1(0)\,\overline{\mathcal{I}^{1-\alpha}_{q,a^-}z_2(0)}\right]
\end{aligned}$$

and

$$\begin{aligned}
\int_0^a u_2\overline{\left({}^c\mathcal{D}^\alpha_{q,0^+}z_1\right)}d_qx &= \int_0^a \left(-\mathcal{D}^\alpha_{q,a^-}u_2\right)\overline{z_1}d_qx \\
-\left[z_1(a)\,\overline{\mathcal{I}^{1-\alpha}_{q,a^-}u_2\left(\frac{a}{q}\right)} - z_1(0)\,\overline{\mathcal{I}^{1-\alpha}_{q,a^-}u_2(0)}\right],
\end{aligned}$$

we get

$$(T_{q,\alpha}u, z) - (u, T_{q,\alpha}z) = [u, z](a) - [u, z](0)\,, \tag{3.4}$$

where $[y, z](x) := y_1(x)\,\overline{\mathcal{I}^{1-\alpha}_{q,a^-}z_2(x)} - \overline{z_1(x)}\mathcal{I}^{1-\alpha}_{q,a^-}y_2(x)$. We proceed to show that the equality $(T_{q,\alpha}u, z) = (u, T_{q,\alpha}z)$ for any $u(.), z(.) \in H$. From the boundary conditions (3.2) and (3.3), we get $[u, z]_a = 0$ and $[u, z]_0 = 0$. Consequently,

$$(T_{q,\alpha}u, z) = (u, T_{q,\alpha}z)\,. \tag{3.5}$$

This completes the proof. □

LEMMA 3.2. *All eigenvalues of the operator $T_{q,\alpha}$ generated by q-FD system defined by (3.1)-(3.3) are real.*

PROOF. Let μ be an eigenvalue with an eigenfunction $z(x)$. From the equality (3.5), we get

$$(T_{q,\alpha}z, z) = (z, T_{q,\alpha}z) = (z, \mu z) = \overline{\mu}(z, z). \tag{3.6}$$

On the other hand,

$$(T_{q,\alpha}z, z) = (\mu z, z) = \mu(z, z). \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\mu(z, z) = \overline{\mu}(z, z), \quad (\mu - \overline{\mu})(z, z) = 0.$$

Since $z \neq 0$, we get $\mu = \overline{\mu}$. □

LEMMA 3.3. *If μ_1 and μ_2 are two different eigenvalues of the operator $T_{q,\alpha}$ generated by q-FD system defined by (3.1)-(3.3), then the corresponding eigenfunctions θ and η are orthogonal.*

PROOF. Let μ_1 and μ_2 be two different real eigenvalues with corresponding eigenfunctions θ and η, respectively. From (3.5), we obtain

$$(T_{q,\alpha}\theta, \eta) = (\theta, T_{q,\alpha}\eta), (\mu_1\theta, \eta) = (\theta, \mu_2\eta)$$
$$(\mu_1 - \mu_2)(\theta, \eta) = 0.$$

Since $\mu_1 \neq \mu_2$, we obtain that $\theta(x)$ and $\eta(x)$ are orthogonal. □

Now let $u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$, $z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in H$. Then, we define the Wronskian of $u(x)$ and $z(x)$ by

$$W(u, z)(x) = u_1(x)\mathcal{I}_{q,a^-}^{1-\alpha} z_2(x) - z_1(x)\mathcal{I}_{q,a^-}^{1-\alpha} u_2(x).$$

THEOREM 3.4. *The Wronskian of any solution of Eq. (3.1) is independent of x.*

PROOF. Let $u(x)$ and $z(x)$ be two solutions of Eq. (3.1). By Green's formula (3.4), we have

$$(T_{q,\alpha}u, z) - (u, T_{q,\alpha}z) = [u, z](a) - [u, z](0).$$

Since $T_{q,\alpha}u = \lambda u$ and $T_{q,\alpha}z = \lambda z$, we have

$$(\lambda u, z) - (u, \lambda z) = [u, z](a) - [u, z](0),$$
$$\left(\lambda - \overline{\lambda}\right)(u, z) = [u, z](a) - [u, z](0).$$

Since $\lambda \in \mathbb{R}$, we have $[u, z](a) = [u, z](0) = W(u, \overline{z})(0)$, i.e., the Wronskian is independent of x. □

COROLLARY 3.5. *If $u(x)$ and $z(x)$ are both solutions of Equation (3.1), then either $W(u, z)(x) = 0$ or $W(u, z)(x) \neq 0$ for all $x \in [0, a]$.*

THEOREM 3.6. *Any two solutions of Equation (3.1) are linearly dependent if and only if their Wronskian is zero.*

Proof. Let $u(x)$ and $z(x)$ be two linearly dependent solutions of equation (3.1). Then, there exists a constant $k > 0$ such that $u(x) = k\ z(x)$. Hence

$$W(u,z) = \begin{vmatrix} u_1(x) & \mathcal{I}^{1-\alpha}_{q,a^-} u_2(x) \\ z_1(x) & \mathcal{I}^{1-\alpha}_{q,a^-} z_2(x) \end{vmatrix} = \begin{vmatrix} kz_1(x) & k\mathcal{I}^{1-\alpha}_{q,a^-} z_2(x) \\ z_1(x) & \mathcal{I}^{1-\alpha}_{q,a^-} z_2(x) \end{vmatrix} = 0.$$

Conversely, the Wronskian $W(u,z) = 0$ and therefore, $u(x) = kz(x)$, i.e., $u(x)$ and $z(x)$ are linearly dependent. ∎

Before proceeding further, we need the following auxiliary functions.

We introduce the function $\phi(x) := \begin{pmatrix} (\mathcal{I}^{\alpha}_{q,a^-}1)(x) \\ (\mathcal{I}^{\alpha}_{q,0^+}1)(x) \end{pmatrix}$. Further, the general solution of the equation $\tau_{q,\alpha}\psi = 0$, i.e.,

$$\begin{pmatrix} 0 & \mathcal{D}^{\alpha}_{q,a^-} \\ {}^{c}\mathcal{D}^{\alpha}_{q,0^+} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

is given by

$$\psi = \begin{pmatrix} \xi_1 \\ \xi_2 \varphi(\alpha,a,x) \end{pmatrix},$$

where

$$\varphi(\alpha,a,x) = \frac{a^{\alpha-1}\left(\frac{qx}{a} : q\right)_{\alpha-1}}{\Gamma_q(\alpha)}. \tag{3.8}$$

Lemma 3.7. *Let*

$$\Delta := c_{11}c_{12} - c_{11}c_{21}$$

and

$$F_\lambda(f) := \{V - \lambda\omega\} f_\lambda, \tag{3.9}$$

where $V(x) := \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}$. *Assume* $\Delta \neq 0$. *Then on the space* $C\left(A^*_{t,\alpha}\right)$, *the q-FD system defined by (3.1)-(3.3) is equivalent to the integral equation*

$$f_\lambda(x) = -MF_\lambda(f) + A(x)T + B(x)Z,$$

where the coefficients M, A, T, B and Z are

$$M := \begin{pmatrix} 0 & \mathcal{I}_{q,0^+}^{\alpha} \\ \mathcal{I}_{q,a^-}^{\alpha} & 0 \end{pmatrix},$$

$$A(x) := \begin{pmatrix} \frac{c_{12}c_{22}}{\Delta} \\ -\frac{c_{21}c_{12}}{\Delta}\varphi(\alpha, a, x) \end{pmatrix},$$

$$T := -\mathcal{I}_{q,a^-}^{\alpha} F_{\lambda 1}(y)\,|_{x=0},$$

$$B(x) := \begin{pmatrix} \frac{c_{12}c_{21}}{\Delta} \\ -\frac{c_{21}c_{11}}{\Delta}\varphi(\alpha, a, x) \end{pmatrix},$$

$$Z := -\mathcal{I}_{q,0^+}^{1} F_{\lambda 2}(y)\,|_{x=a},$$

and the function $\varphi(\alpha, a, x)$ *is defined in (3.8).*

Proof. Using fractional composition rules and (3.9), we can rewrite the equation (3.1) as follows:

$$\tau_{q,\alpha}\left[f_\lambda(x) + MF_\lambda(f)\right] = 0.$$

Thus, we get

$$f_\lambda(x) + MF_\lambda(f) = \begin{pmatrix} \xi_1 \\ \xi_2\varphi(\alpha, a, x) \end{pmatrix},$$

i.e.,

$$f_\lambda(x) = -MF_\lambda(f) + \begin{pmatrix} \xi_1 \\ \xi_2\varphi(\alpha, a, x) \end{pmatrix}. \tag{3.10}$$

Now, we shall connect the coefficients ξ_i $(i = 1, 2)$ to the values c_{ij} $(i, j = 1, 2)$ in the boundary conditions (3.2)-(3.3). From the equation (3.10), we obtain

$$Kf_\lambda(x) = -KMF_\lambda(f) + K\begin{pmatrix} \xi_1 \\ \xi_2\varphi(\alpha, a, x) \end{pmatrix},$$

where $K := \begin{pmatrix} 0 & \mathcal{I}_{q,a^-}^{1-\alpha} \\ 1 & 0 \end{pmatrix}$. Then we have

$$\begin{pmatrix} \mathcal{I}_{q,a^-}^{1-\alpha} f_{\lambda 2} \\ f_{\lambda 1} \end{pmatrix} = -\begin{pmatrix} \mathcal{I}_{q,a^-}^{1} & 0 \\ 0 & \mathcal{I}_{q,0^+}^{\alpha} \end{pmatrix} F_\lambda(f) + \begin{pmatrix} \mathcal{I}_{q,a^-}^{1-\alpha}\left[\xi_2\varphi(\alpha, a, x)\right] \\ \xi_1 \end{pmatrix},$$

i.e,

$$\begin{pmatrix} \mathcal{I}_{q,a^-}^{1-\alpha} f_{\lambda 2} \\ f_{\lambda 1} \end{pmatrix} = \begin{pmatrix} -\mathcal{I}_{q,a^-}^{1} F_{\lambda 1}(f) \\ -\mathcal{I}_{q,0^+}^{\alpha} F_{\lambda 2}(f) \end{pmatrix} + \begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix}.$$

By virtue of (3.2) and (3.3), we conclude that

$$\begin{aligned} f_{\lambda 1}(0) &= \xi_1, \\ f_{\lambda 1}(a) &= -\mathcal{I}^{\alpha}_{q,0^+} F_{\lambda 2}(y)\mid_{x=a} + \xi_1, \\ \mathcal{I}^{1-\alpha}_{q,a^-} f_{\lambda 2}(0) &= -\mathcal{I}^{1}_{q,a^-} F_{\lambda 1}(y)\mid_{x=0} + \xi_2, \\ \mathcal{I}^{1-\alpha}_{q,a^-} f_{\lambda 2}\left(\frac{a}{q}\right) &= \xi_2. \end{aligned}$$

This leads to the system of equations

$$c_{11}\xi_1 + c_{12}\xi_2 = -c_{12},\ Tc_{21}\xi_1 + c_{22}\xi_2 = -c_{21}Z.$$

Since $\Delta \neq 0$, the solutions for coefficients $\xi_j, j = 1, 2$ is unique:

$$\begin{aligned} \xi_1 &= \frac{c_{12}(c_{21}Z - c_{22}T)}{\Delta}, \\ \xi_2 &= \frac{c_{21}(c_{12}T - c_{11}Z)}{\Delta}. \end{aligned}$$

We have finished the proof of the lemma. □

Now, we prove the existence and uniqueness of eigenfunction of the regular q-FD system defined by (3.1)-(3.3). In the next result, we use the following notations:

$$A := \|A(x)\|_C,\ B := \|B(x)\|_C,\ S_\phi := \|\phi(x)\|_C,$$

where $\|.\|_C$ denotes the supremum norm on the space $C\left(A^*_{t,\alpha}, E\right)$.

Theorem 3.8. *Let $\alpha \in (0,1)$ and assume $\Delta \neq 0$. Then unique continuous function y_λ for the regular q-FD system defined by (3.1)-(3.3) corresponding to each eigenvalue obeying*

$$\|V - \lambda\omega\|_C \leq \frac{1}{S_\phi + A\|\phi(a)\|_C + Ba} \tag{3.11}$$

exists and such eigenvalue is simple.

Proof. Let us define the mapping $L : C\left(A^*_{t,\alpha}, E\right) \to C\left(A^*_{t,\alpha}, E\right)$ by

$$Lf := -MF_\lambda(f) + A(x)T + B(x)Z.$$

Now, we show that the equation (3.1) can be interpreted as a fixed point condition on the space $C\left(A^*_{t,\alpha}, E\right)$. Using the following estimate

$$\|F_\lambda(g) - F_\lambda(h)\|_C \leq \|g - h\|_C \|V - \lambda\omega\|_C,$$

we conclude that

$$\begin{aligned} \|Lg-Lh\|_C &\leq \|g-h\|_C \|V-\lambda\omega\|_C S_\phi + A\|g-h\|_C \|\phi(a)\|_C \\ &\quad + Ba\|g-h\|_C \|V-\lambda\omega\|_C \\ &= \|V-\lambda\omega\|_C \|g-h\|_C (S_\phi + A\|\phi(a)\|_C + Ba) \\ &= \Pi\|g-h\|_C, \end{aligned}$$

where $\Pi = \|V-\lambda\omega\|_C (S_\phi + A\|\phi(a)\|_C + Ba)$. By the condition (3.11), the mapping L is a contraction on the space $C\left(A^*_{t,\alpha}, E\right)$ so it has a unique fixed point. Therefore, such eigenvalue is simple. □

CONCLUSION 3.9. In this paper, we study regular q-fractional Dirac systems. In this context, we investigate the properties of the eigenvalues and the eigenfunctions of this system. Finally, we give a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions.

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q-frakcijski sustavi Diracovog tipa

Bilender P. Allahverdiev i Hüseyin Tuna

Sažetak. Ovaj članak je posvećen proučavanju regularnih q-frakcijskih sustava Diracovog tipa. Proučavaju se svojstva svojstvenih vrijednosti i svojstvenih funkcija tih sustava. Korištenjem teorema o fiksnoj točki, daje se dovoljan uvjet na svojstvene vrijednosti za postojanje i jedinstvenost pridruženih svojstvenih funkcija.

Bilender P. Allahverdiev
Department of Mathematics
Faculty of Arts and Sciences,
Süleyman Demirel University
32260 Isparta, Turkey
E-mail: bilenderpasaoglu@sdu.edu.tr

Hüseyin Tuna
Department of Mathematics
Faculty of Arts and Sciences
Mehmet Akif Ersoy University
15030 Burdur, Turkey
E-mail: hustuna@gmail.com

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